$L(2,1)$-Labeling of Oriented Planar Graphs (Extended Abstract)

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Abstract

In this paper we study the $L(2,1)$-labeling problem on oriented planar graphs with particular attention on the subclasses of oriented prisms, Halin and cactus graphs. For these subclasses more accurate results are presented.

keywords: $L(2,1)$-labeling, oriented graph coloring, digraphs, prisms, Halin graphs, cacti.

1 Introduction and preliminaries

The $L(2,1)$-*labeling* of a graph G is a function l from the vertex set $V(G)$ to the set of all nonnegative integers such that :
– $|l(x) - l(y)| \geq 2$ if x and y are adjacent in G, and
– $|l(x) - l(y)| \geq 1$ if x and y are at distance 2 in G.

This problem has been introduced first by Griggs and Yeh [5] as a variation of the frequency assignment problem of wireless networks. A natural extension, recently introduced [2], is the $L(2,1)$-labeling on *directed graphs.* The definition is the same as in the non directed case in which the distance between vertices x and y is defined as the length of the shortest directed path from x to y. In agreement with the non directed case, the $L(2,1)$-labeling number $\vec{\lambda}(D)$ of a digraph D, is the smallest number μ such that D has an $L(2,1)$-labeling with $\max\{l(v) : v \in V(D)\} = \mu$. By extension, for a class $\mathcal{C}$ of digraphs, we denote by $\vec{\lambda}(\mathcal{C})$ the maximum $\vec{\lambda}(D)$ over all $D \in \mathcal{C}$. In this paper we will consider the $L(2,1)$-labeling problem only on digraphs that are orientations of finite simple graphs (i.e. graphs without loops or multiple edges). A related concept to the $L(2,1)$-labeling of oriented graphs is the oriented chromatic number. An *(oriented) coloring* of $\vec{G} = (V, A)$ is a function $f : V \to \mathbb{N}$ such that

- $f(u) \neq f(v)$ if $(u, v) \in A$ and

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- for any two arcs (u, v) and (x, y), the colors assigned to x and v and to y and u cannot be the same.

The *oriented chromatic number* of $\vec{G}$, $\vec{\chi}(\vec{G})$, is the smallest integer κ such that the oriented graph $\vec{G}$ has a coloring using κ colors. It is not difficult to prove the following result (cf. [4]):

Proposition 1 *If $\vec{G}$ is an orientation of a graph G, then*

$$2\big(\chi(G) - 1\big) \leq \vec{\lambda}(\vec{G}) \leq 2\big(\vec{\chi}(\vec{G}) - 1\big).$$

It is worth mentioning that, in general, this is a quite unsatisfactory result as the oriented chromatic number of a digraph and the chromatic number of its underlying graph can be far from each other. However in the case of oriented planar graphs this result turns out to be quite useful. Indeed, combining Proposition 1 with the results in [6, 7] that upper bound the oriented chromatic number of a planar graph, we obtain the following:

Proposition 2 *Given an oriented planar graph $\vec{G}$ with girth g, $\vec{\lambda}(\vec{G})$ is upper bounded as follows:*

- *If $g \geq 16$ then $\vec{\lambda}(\vec{G}) \leq 8$.*
- *If $g \geq 11$ then $\vec{\lambda}(\vec{G}) \leq 12$.*
- *If $g \geq 7$ then $\vec{\lambda}(\vec{G}) \leq 22$.*
- *If $g \geq 6$ then $\vec{\lambda}(\vec{G}) \leq 62$.*
- *otherwise $\vec{\lambda}(\vec{G}) \leq 158$.*

2 Subclasses of oriented planar graphs

In this section we prove some results concerning $\vec{\lambda}$ for some particular subclasses of planar graphs. Due to space limitations almost all the proofs are omitted.

A ***planar prism graph*** Pr_n is a graph isomorphic to $C_n \times P_2$. The $L(2,1)$-labeling number of an unoriented prisms is well-known. The following theorem proves that the same result holds in the oriented case.

Theorem 1 *Let $\mathcal{P}_n$ be the set of all the orientations of the planar prism graph Pr_n. Then $\vec{\lambda}(\mathcal{P}_n) = 5$ if $n \equiv 0 \mod 3$ and $\vec{\lambda}(\mathcal{P}_n) = 6$ otherwise.*

The proof follows by showing that for any prism Pr_n there exists an orientation $\vec{Pr}_n$ which preserves the two length distances between vertices.

A ***Halin graph*** H is a planar graph constructed from a plane embedding of a tree with at least four vertices and with no vertices of degree 2, by connecting all the leaves with a cycle that passes around the tree in the natural cyclic order defined by the embedding of the tree. An *n-wheel* W_n is a Halin graph whose tree is a star.

Theorem 2 *Let $\mathcal{W}_n$ be the set of all the orientations of the n-wheel. It holds* $8 \leq \vec{\lambda}(\mathcal{W}_n) \leq 9$.

Proof. In all the forthcoming theorems the lower bound follows by exhibiting a graph for which $\vec{\lambda}$ attains the value claimed. We omit these proofs and sketch only the proof of the upper bound. Consider an arbitrary oriented wheel $\vec{W}_n$. Without loss of generality, label the center vertex c with 0. Now, consider a set of labels $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_4$ (to be defined later) none of them containing 0. We label the vertices of the cycle according to these sets. Observe that any vertex in the out-neighborhood of c is connected by a dipath of length two to every vertex in the in-neighborhood of c. Thus the sets of labels of these two neighborhoods must be disjoint. Let $v_0, \ldots, v_{n-1}$ be an ordering of the vertices on the cycle of $\vec{W}_n$ such that v_i is adjacent to v_{i+1}, with $0 \leq i \leq n-2$ and v_0 is adjacent to v_{n-1}. In analogy with the usual (unoriented) $L(2,1)$-labeling of a cycle [5], we label the vertices of the cycle using the colors $\mathbf{a}_i$ according to the value of $n \mod 3$ (see Fig. 1(a)). We define the sets $\mathbf{a}_i$ as the following classes: $\mathbf{a}_0 = \{2,3\}$, $\mathbf{a}_1 = \{4\}$, $\mathbf{a}_2 = \{5,6\}$,

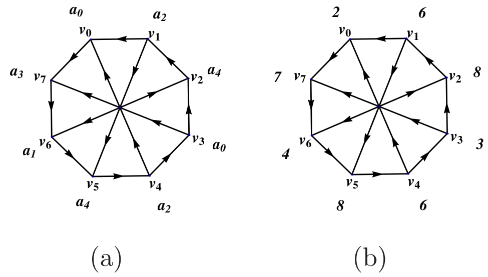

Figure 1: Labeling of an oriented wheel according to: (a) the cycle (b) the tree

$\mathbf{a}_3 = \{7\}$ and $\mathbf{a}_4 = \{8,9\}$. We assign to each vertex labeled $\mathbf{a}_i$ respectively the minimum or maximum element of the set $\mathbf{a}_i$ according to the fact that the vertex belongs to the out- or in-neighborhood of the center c (see Fig. 1(b)). Finally, observe that there is at most one vertex labeled $\mathbf{a}_1$ and at most one vertex labeled $\mathbf{a}_3$. Thus, we transform the label i in the unique element of $\mathbf{a}_i$. It is not difficult to see that this is a feasible $L(2,1)$-labeling of $\vec{W}_n$. ∎

Using almost the same technique as in the previous theorem we prove the following:

Theorem 3 *Let $\mathcal{H}$ be the set of all the oriented Halin graphs, then we have* $8 \leq \vec{\lambda}(\mathcal{H}) \leq 10$.

A ***cactus*** is a connected graph in which any two simple cycles have at most one vertex in common.

Theorem 4 *Let $\mathcal{Y}$ be the set of all the oriented cacti then* $6 \leq \vec{\lambda}(\mathcal{Y}) \leq 8$.

The proof follows by showing that there is a homomorphism from any cactus $\vec{Y}$ to some tournament on 5 vertices.

3 Concluding remarks and open problems

In this paper we approach the problem of determining $\vec{\lambda}$ for oriented planar graphs. Furthermore when focusing on some particular subclasses of palanr graphs we provide nearly tight bounds for $\vec{\lambda}$. However, it is clear that many problems remain open and there is room for many other interesting questions on this topic.

In particular it is worth to note that in the unoriented case there is a strong relationship between λ and the graph's maximum degree Δ, and thus it efficiently expresses the value of λ. However the $L(2,1)$-labeling problem on oriented graphs presents different issues with respect to the unoriented case and Δ is not an appropriate parameter anymore.

Due to the fact that very few classes of oriented graphs have been investigated, it is not possible yet to identify the most natural graph parameter that efficiently expresses the value of $\vec{\lambda}$ for arbitrary oriented graphs. However, for the class of oriented planar graphs it seems reasonable to suggest that the girth of the underlying graph is in some relation with $\vec{\lambda}$. This is suggested by Proposition 2 and the following Conjecture in [1]

Conjecture 1 *Every oriented planar graph D whose underlying graph has girth $g \geq 5$ has $\vec{\lambda}(D) \leq 5$.*

Nevertheless; this relation cannot be as strong as in the unoriented case between λ and Δ. Thus, investigating in this direction is an interesting open problem.

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